Calculate the limit
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$$\lim_{x\to 5} \frac{x^x - x^5}{x^x - 5^x}.$$

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We will prove that $\lim_{x\to a} \frac{x^x - x^a}{x^x - a^x} = \ln a$ for $\forall a > 0$.

We have

$$\lim_{x\to a} \frac{x^x - x^a}{x^x - a^x} = \lim_{x\to a} \frac{x^x(x^{a-x} - 1)}{x^x((a/x)^x - 1)} = \lim_{x\to a}\left(\frac{e^{(a-x)\ln x} - 1}{(a-x)\ln x} \cdot \frac{x\ln(a/x)}{e^{x\ln(a/x)} - 1} \cdot \frac{(a-x)\ln x}{x\ln(a/x)}\right) =$$

$\lim_{x\to a} \frac{(a-x)\ln x}{x\ln(a/x)} = \ln a \lim_{x\to a} \frac{\frac{a}{x} - 1}{\ln(a/x)} = \ln a$ because $\lim_{x\to a} \frac{e^{(a-x)\ln x} - 1}{(a-x)\ln x} = \lim_{t\to 0} \frac{e^t - 1}{t} = 1$

($t := (a-x)\ln x$ and $\lim_{x\to a}(a-x)\ln x = 0$), $\lim_{x\to a} \frac{e^{x\ln(a/x)} - 1}{x\ln(a/x)} = \lim_{t\to 0} \frac{e^t - 1}{t} = 1$

($t := x\ln(a/x)$ and $\lim_{x\to a} x\ln(a/x) = 0$) and $\lim_{x\to a} \frac{\ln(a/x)}{a/x - 1} = \lim_{t\to a} \frac{\ln(1+t)}{t} = 1$

($t := a/x - 1$ and $\lim_{x\to a}(a/x - 1)$).